Two cut-and-glue rules

A cut-and-glue rule alters a certain configuration of I-tiles by cutting and glueing them in a way that results in new I-tiles exactly tiling the same region as in the initial configuration.

The configuration for the first rule is as in the left hand side of Figure 1 with the conditions:

1. The shortest tile above the horizontal cross-bar that spans the pattern must have integer height, h say, that is, it is a V-tile; and

2. none of the vertical edges of tiles meet on the cross-bar except at the two ends of the cross-bar.
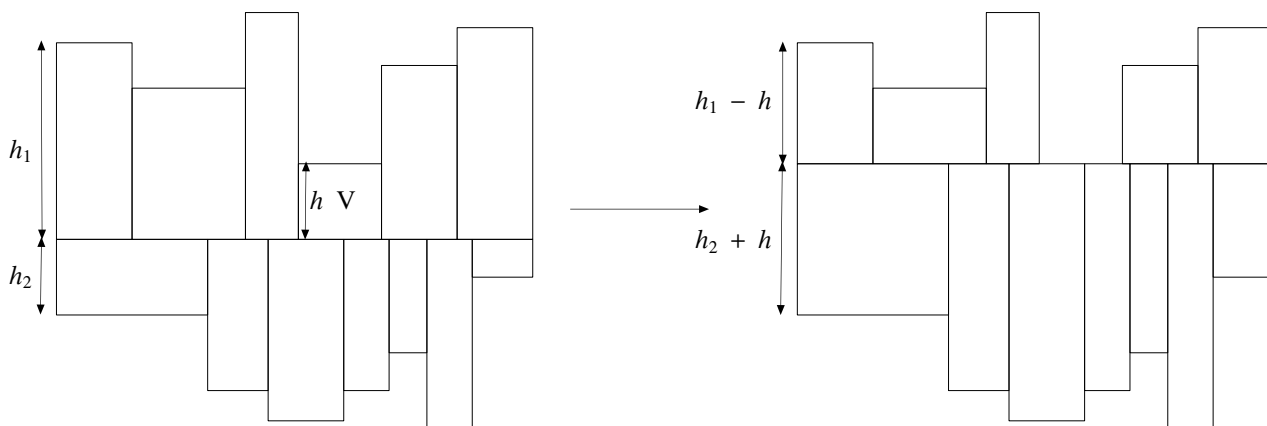

Figure 1

The cross-bar is raised by h units and the new tiles occupy the same region as before. Further, H-tiles, respectively V-tiles, whether shortened above the cross-bar or lengthened under the cross-bar, remain H-tiles, respectively V-tiles. So all the transformed tiles are still I-tiles and there is at least one less I-tile (only one less in this diagram).

There is the upside-down version of this rule if the shortest tile under the cross-bar is a V-tile, in which case the cross-bar is lowered by the height of this shortest tile. Again, the new tiles are I-tiles and there is at least one less tile. There are also two sideways version of the rule, where the cross-bar is vertical with the affected tiles on the left and the right of the cross-bar (rotate the above two configurations through 90°) and the tile of least width on the right (respectively, left) of the cross-bar is of integer width, w. In this case the cross-bar moves right (respectively, left) by w units, again resulting in at least one less I-tile.

Note that, if a configuration of tiles occurs as above, but where one or more pairs of edges meet at internal point(s) of the cross-bar, then the configuration can be split up into two or more (adjacent) configurations each of which satisfies condition 2 above. Then the cut-and-glue rule can be applied to any of these for which condition 1 holds.

Neither of the I-tilings below can be reduced to a single tile by any of the four orientations of the cut-and-glue rule.
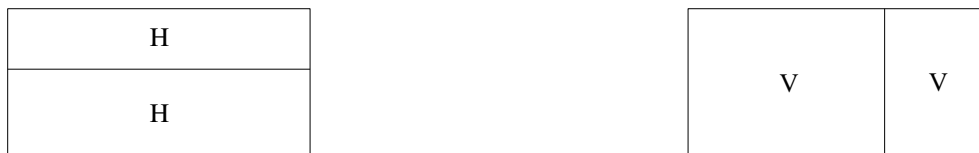

Figure 2

So we introduce an extra 'merge' rule, where two tiles with a common edge are glued along that edge. (Figure 3, next page). It is easy to check that, for 2 I-tiles, the result is an I-tile even when one is a V-tile and the other an H-tile.
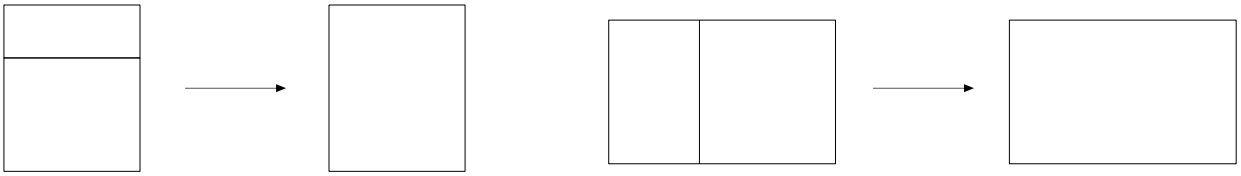
Figure 3