M337 A1 Complex Numbers: Exercises

1. Let $z = 1 + i$.
   (a) Find $\bar{z}$, $z\bar{z}$ and z^{-1} in Cartesian form.
   (b) By writing z in polar form, find z^2, $\sqrt{z}$ and z^{-1} in polar form.
   (c) What is the smallest positive integer value of n for which z^n is real?
   (d) By finding $(-\sqrt{3} + i)/(1 + i)$ in both Cartesian and polar forms, verify that
$$\sin\left(\frac{7\pi}{12}\right) = \frac{1 + \sqrt{3}}{2\sqrt{2}} \qquad \text{and} \qquad \cos\left(\frac{7\pi}{12}\right) = \frac{1 - \sqrt{3}}{2\sqrt{2}}.$$

2. Use de Moivre's Theorem for the following problems.
   (a) Find $(1 - \sqrt{3}i)^{10}$ in Cartesian form.
   (b) Show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

3. (a) If ω is a complex cube root of unity, plot ω and ω^2 in the complex plane and show that $1 + \omega + \omega^2 = 0$.
   (b) Determine all the fifth roots of $i/32$ and plot them in the complex plane.

4. Sketch each of the following sets in the complex plane, using the conventions on p.38 of *Unit A1*.
$$A = \{z : |z - 1 - i| = \sqrt{2}\} \qquad B = \{z : 1 \leqslant |z - i| \leqslant 2\}$$
$$C = \{z : -\pi/3 < \operatorname{Arg} z < \pi/3\} \qquad D = \{z : \operatorname{Arg} z < \pi\}$$
$$E = \{z : \operatorname{Re} z + \operatorname{Im} z < 1\} \qquad F = \{z : |z| < |z - 1 - i|\}$$
$$G = \{z : \operatorname{Arg}(z - i) = \pi\} \qquad H = \{i/n : n \in \mathbb{N}\}$$
   [These sets will be used again in the A3 exercises sheet.]

5. (a) For what complex numbers z_1 and z_2 is it true that $|z_1 + z_2| = |z_1| + |z_2|$?
   (b) Find an upper bound for the set:
$$\left\{\left|\frac{2z - 11i}{2z^2 + 11}\right| : |z| = 2\right\}.$$

6. A basic fact used in Number Theory is: If two integers can be expressed as a sum of two squares, then so can their product. Prove this result as follows. Let $M = a^2 + b^2$ and $N = c^2 + d^2$, where a, b, c, d are integers. By considering $|(a + ib)(c + id)|^2$, prove that MN can be expressed as a sum of two squares.

7. Using a picture, show that, for fixed $a, b \in \mathbb{C}$, $a \neq b$, $|z - a| = |z - b|$ is the equation of a line.

8. Let $a, b, c, d \in \mathbb{C}$ be four points on the unit circle such that $a + b + c + d = 0$. Show that the points must be vertices of a rectangle. [If stuck, try the case $\operatorname{Im} a = \operatorname{Im} b$ first.]

9. Show geometrically that if $|z| = 1$, $z \neq -1$, then $\operatorname{Im}\left[z/(z + 1)^2\right] = 0$. Apart from the unit circle, what other points satisfy this equation?

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