M208 I1: A Periodic Non-Trigonometric Function

A periodic function is not necessarily a trigonometric function. There are simpler examples of periodic non-trigonometric functions such as $f(x) = x - [x]$, where $[x]$ is the integer part of x, than the following example; but the example below is not only continuous but differentiable (i.e. smooth) and has a graph almost indistinguishable from the graph of $y = \sin(\pi x)$.

The example can be most clearly expressed as a composite of two real functions.

Define $h : \mathbb{R} \to \mathbb{R}$ by

$$x \mapsto x - 2[(x+1)/2]$$

and $g : \mathbb{R} \to \mathbb{R}$ by

$$u \mapsto 4u\left(1 - |u|\right).$$

Then let $f : \mathbb{R} \to \mathbb{R}$ be $f = g \circ h$.

So $f(x) = 4\left(x - 2[(x+1)/2]\right)\left(1 - \left|x - 2[(x+1)/2]\right|\right)$, but this is not helpful in understanding f. To understand the function f, first check that $h(x) = x$, for $-1 \leqslant x < 1$, that $h(\mathbb{R}) = [-1,1)$ and that h is periodic with period 2. Second, you only need to sketch g for the range $-1 \leqslant u < 1$ and check that you get the part of the graph in Figure 1 from -1 to 1.

Figure 1 shows the graph of f.

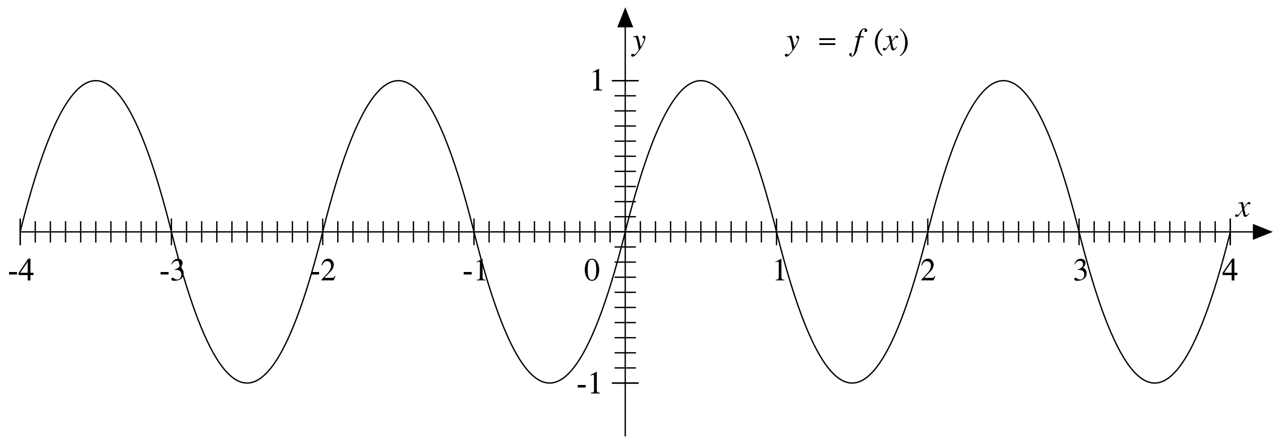

Figure 1. Graph of $y = f(x)$.

For a comparison with the function $\sin(\pi x)$ we have the following. (Figure 2)

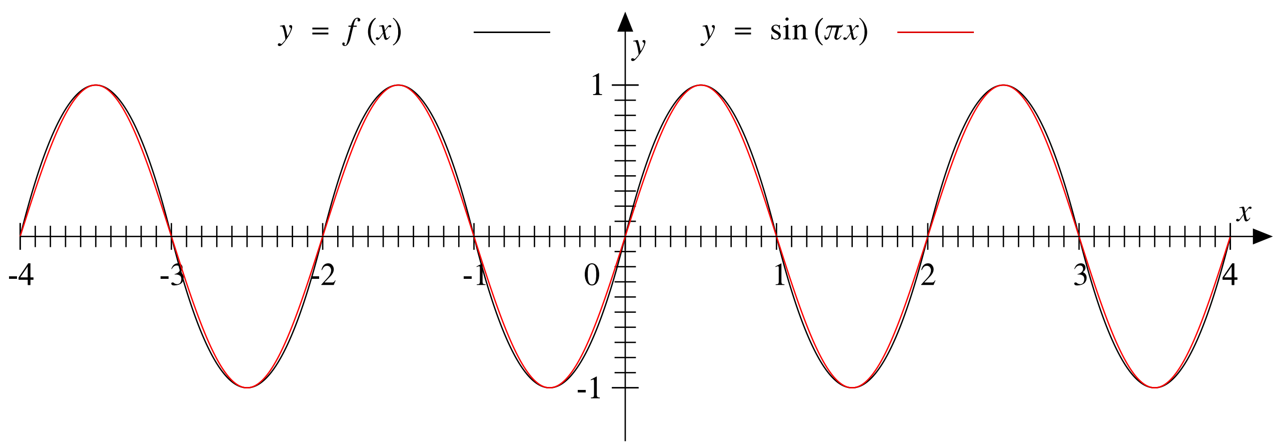

Figure 2. Graphs of $y = f(x)$ and $y = \sin(\pi x)$.

END