Family of Cassini Curves and the Orthogonal Family of Curves

A Cassinian curve is the locus of a point for which the product of its two distances from two fixed points is a constant. If ‘product’ is replaced by ‘sum’ in this definition then we would have an ellipse. Seven years before Isaac Newton published his *Principia* in 1687 in which he showed that planets orbit the Sun in ellipses, Giovanni Cassini had instead proposed that these orbits were curves for which the product of the distances from the focuses was constant. [Tristan Needham, 1997, "*Visual Complex Analysis*", pp60-63]

In the diagram below, the two fixed points, also known as focuses, are at $(1,0)$ and $(-1,0)$. In black are the Cassinian curves for various values of the product of the two distances from the focuses and in red is the Cassinian curve, a figure of eight on its side, for which the product is 1. In general, the figure eight Cassini curve occurs when the product of the two distances from the focuses is the square of half the distance apart of the focuses. When the product of distances is greater than 1, the Cassini curve is a simple loop outside the figure of eight. For values of the product less than 1, the Cassini curve becomes two loops inside the figure of eight, each encircling one focus. Notice that for larger values of the product the curves look increasingly like ellipses.

In green is the family of curves orthogonal to the Cassini curves, that is, each green curve cuts each Cassini curve at right angles (defined as the two tangents at the point of intersection being at right angles to each other). Despite the Cassini curves having quartic equations, the orthogonal family have quadratic equations being a family of central rectangular hyperbolas all passing through the two focuses, including the ‘degenerate hyperbola’ consisting of the two axes.